Solar Physics & Space Plasma Research Centre (SP^{2}RC)

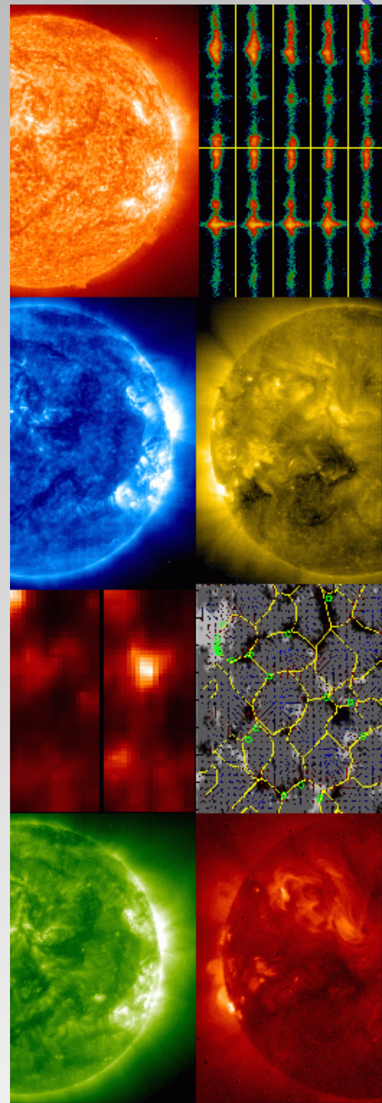
Fundamentals of MHD in Space Research I

Robert Erdélyi

Robertus@sheffield.ac.uk

SP^{2}RC, School of Mathematics and Statistics,
The University of Sheffield (UK)

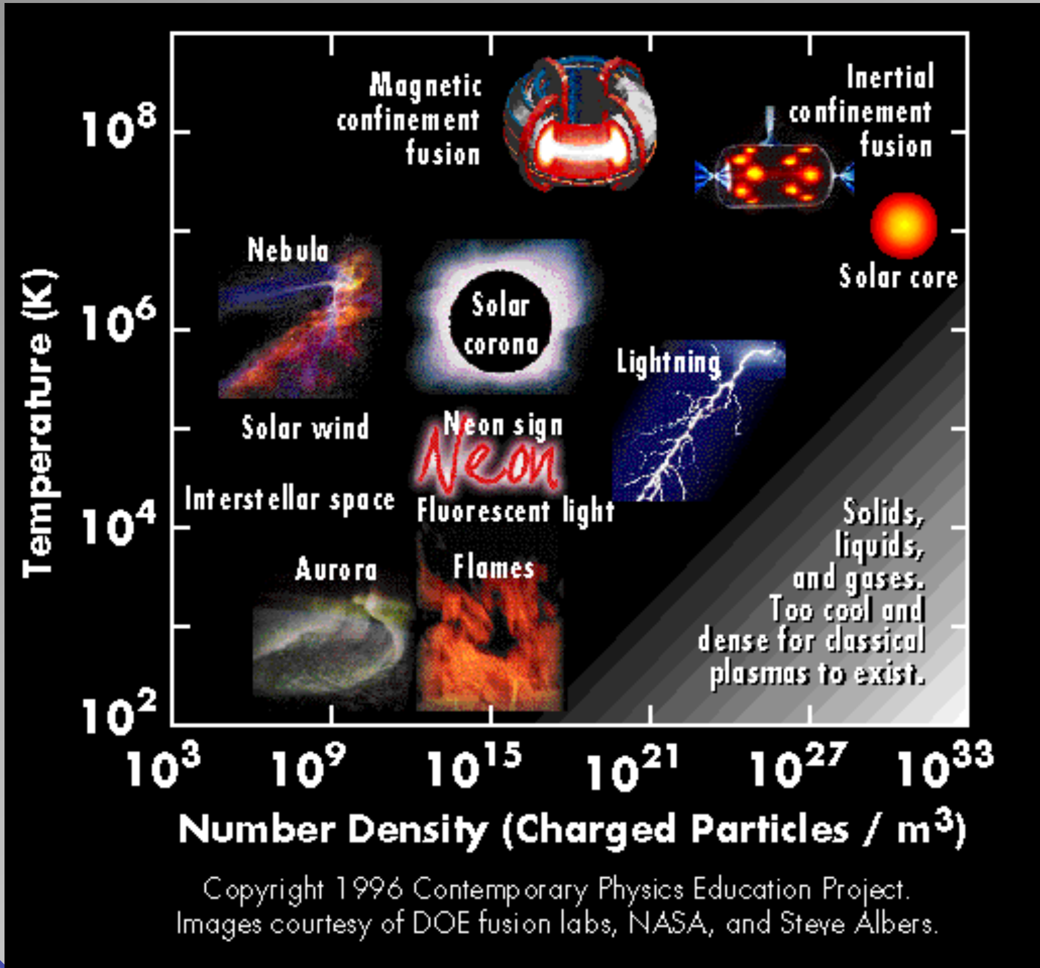
http://robertus.staff.shef.ac.uk



The Outline

- Introduction
- Magnetic Sun
- MHD equations
- Potential and force-free fields
- Selected applications (sunspots, prominences)
- Conclusions

Why Bother?



Modern Physics → Plasma Physics

Solar and stellar interiors are composed of *ionised plasma* – and hence are excellent conductors of electricity

In fact, 99.9% matter of Universe is in plasma state!

ST system is a natural plasma laboratory →geo-, astro- and tokamak physics,

To explore space plasmas waves, in general, are excellent diagnostic tools!!!

Why bother: "Big questions"

• What is the basis of stability and dynamics of solar atmospheric and ST structures?

• What mechanisms are responsible for heating in the solar atmosphere up to several million K?

• What accelerates the solar wind up to measured speeds exceeding 700 km/s?

• What are the physical processes behind the enormous energy releases (e.g. solar flares, megnetospheric substorms, energisation of ULF waves)?

What is the MHD model?

- Single fluid (continuum) approximation, macroscopic description
- Locally charged, globally neutral "close to" LTE
- MHD: perturbations of magnetic field, plasma velocity and plasma mass density, described by the MHD ("single fluid" approximation) set of equations, which connects the magnetic field B, plasma velocity v, kinetic pressure p and density ρ.
- Simplified Maxwell's eqs + "classical" fluid dynamics

Why study MHD?

MHD plays a crucial approximation in the description of dynamics and structure of the solar interior, the entire solar atmosphere (sunspots, chromosphere, TR, corona, solar wind) and in Earth' magnetosphere. MHD approximation is adequately describes

• the evolution and development of plasma perturbations,

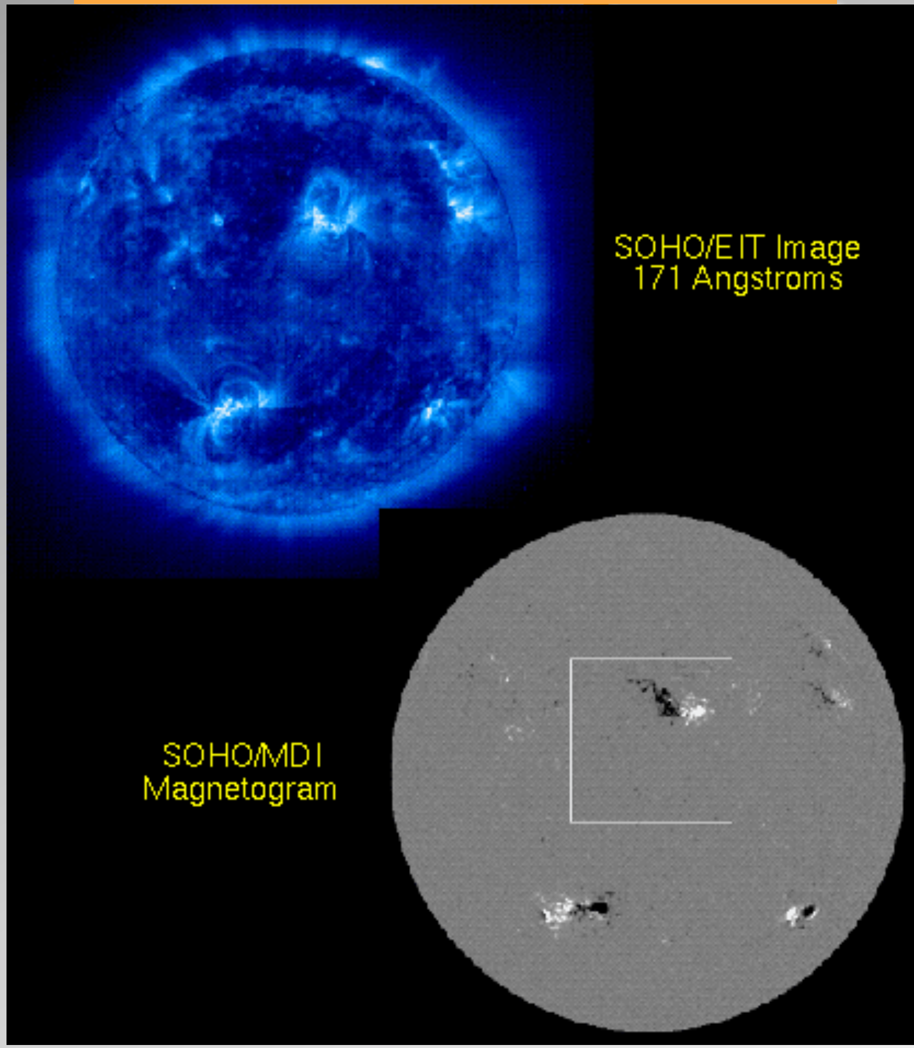
• the transfer of plasma energy and momentum,

• plasma heating / acceleration,

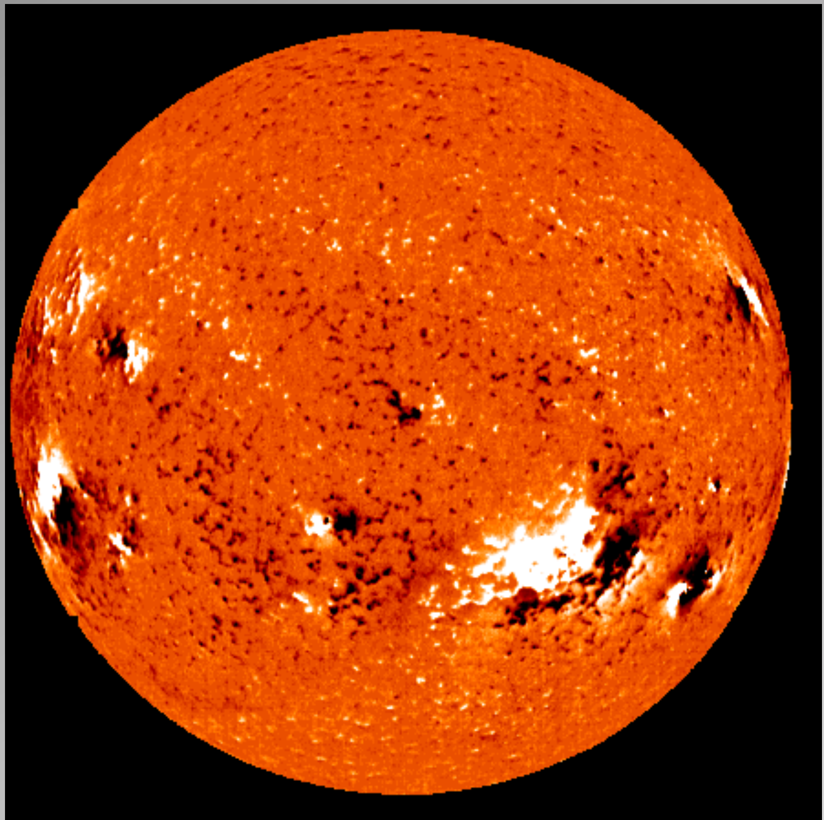
• helioseismology, solar atmospheric (magneto) seismology, magnetosphere seismology.

• Also, we use it because it is relatively simple when compared to other approaches (e.g., kinetic theory)!

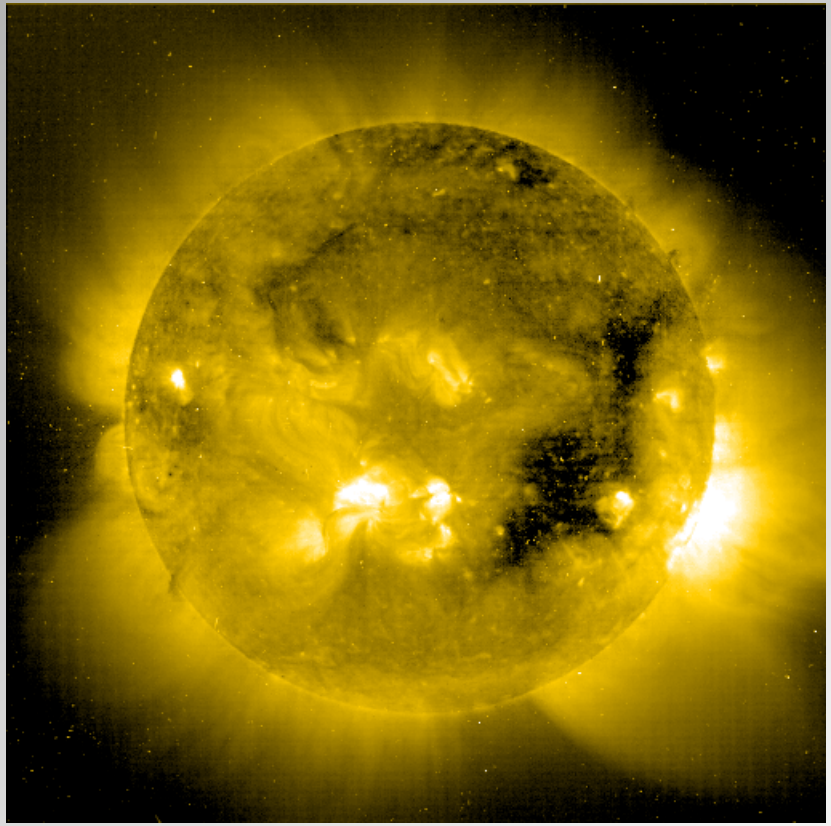
Is the Sun magnetic?



Is the Sun magnetic?

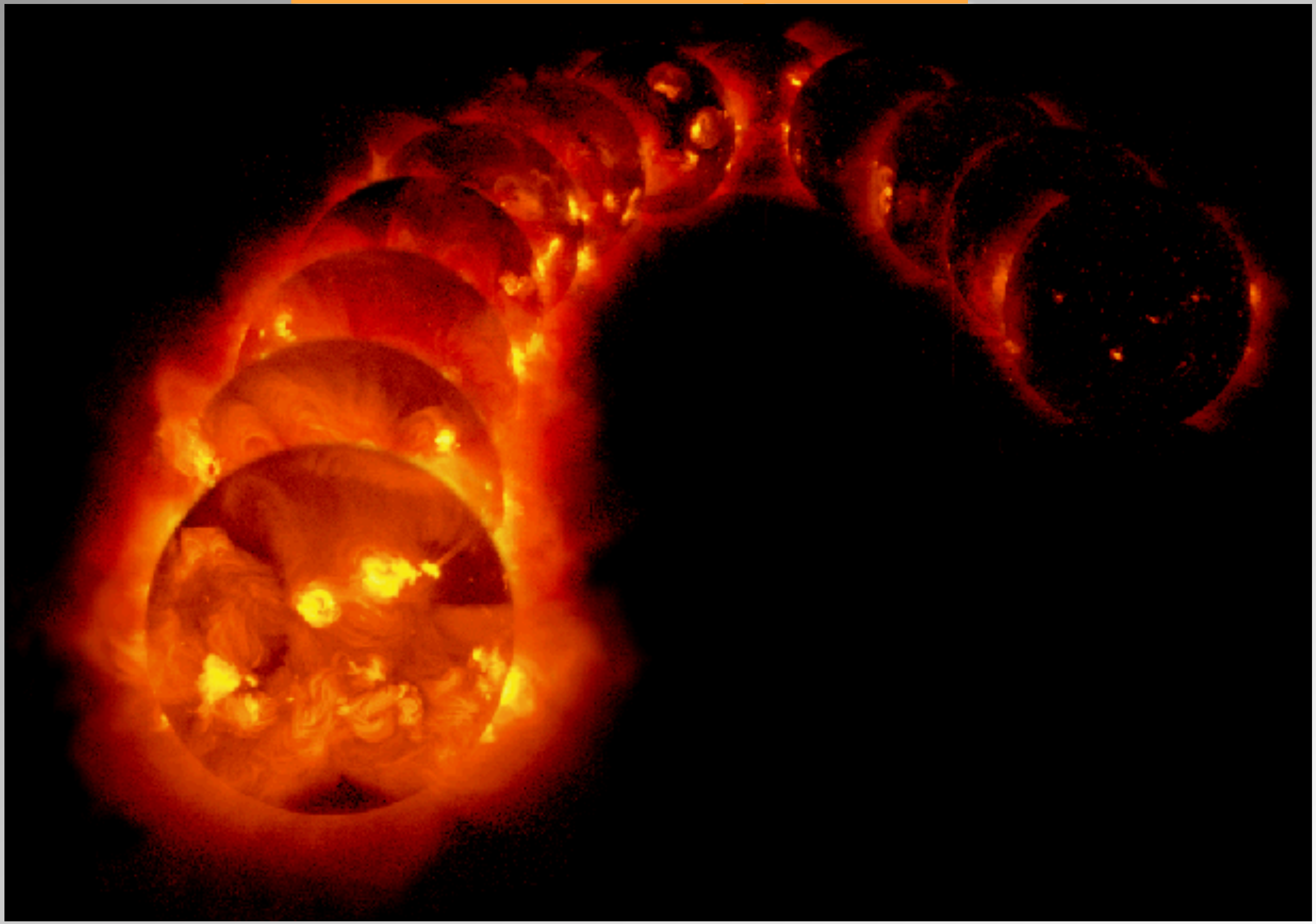


Ca II emission

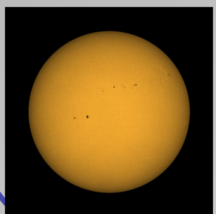


Extreme ultra-violet

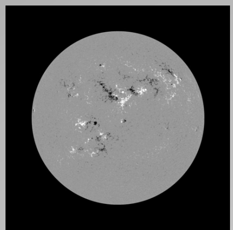
Is the Sun magnetic?



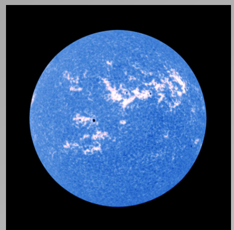
Magnetic coupling: the dynamic Sun



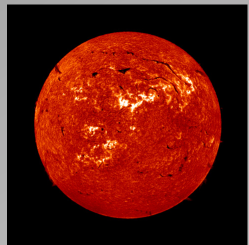
Visible
5000 K



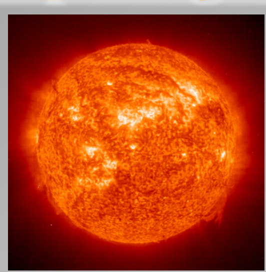
Magnetic field
5000 K



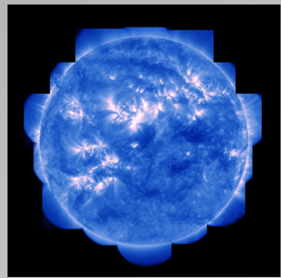
UV 1600 Å
8000 K



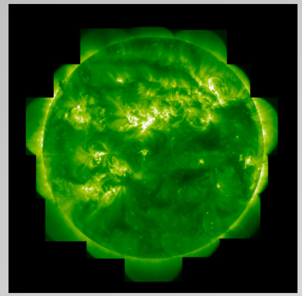
Hα
15,000 K



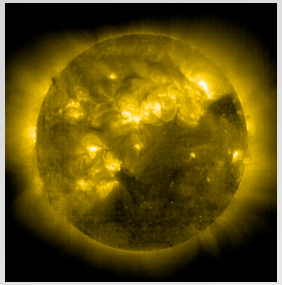
He EUV
50,000 K



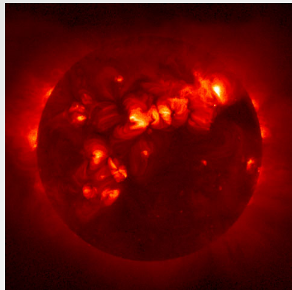
Fe VIII/IX EUV
1 MK



Fe XI
1.5 MK



Fe XIV
3 MK



X rays
4-6 MK

- Photosphere – chromosphere – TR – corona are **<u>magnetically coupled</u>**.
- Very highly **structured** and **dynamic**; challenge for seismology

Magnetic coupling: dynamic STS

• Photosphere – chromosphere – TR – corona (inluding solar wind) – magnetosphere – Earth's upper atmosphere are all **magnetically coupled**.

• Very highly **structured** and **dynamic**.

MHD seismology is a perfect tool to study this coupled, dynamic an structured system.

Two (biassed) particularly exciting aspects:

• **Influence of atmosphere** on global oscillations.

• **Role of *p* modes in the dynamics of the atmosphere!** (Not yet explored.)

Governing equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad (1.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{j} \times \mathbf{B} + \mathbf{F}_{v} \qquad (1.2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad (1.3)$$

$$\rho T \left(\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s \right) = -\mathcal{L} \qquad (1.4)$$

Governing equations

Notation: ρ is density, **v** velocity, p pressure, **B** magnetic induction, **E** electric field, **j** electric current, **T** temperature, s entropy per unit mass, $\mathbf{F}_\nu$ viscosity force, and L energy loss function

Ampere's law:

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B} \qquad (1.5)$$

μ is magnetic permeability of empty space

Viscous force in isotropic plasmas $(\omega_i \tau_i << 1)$:

$$\mathbf{F}_\nu = \nu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla \nabla \cdot \mathbf{v} \right) \qquad (1.6)$$

ν is kinematic viscosity, ρν = const is dynamic viscosity

Governing equations

Ohm's law:

$$\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \mathbf{j} + \frac{m_i\sigma}{\rho e}\mathbf{j}\times\mathbf{B} \qquad (1.7)$$

σ is conductivity, m_i ion mass, e proton electric charge. Last term is Hall current.

Clapeiron law (R gas constant, $\tilde{\mu}$ mean atomic weight):

$$p = (R/\tilde{\mu})\rho T \qquad (1.8)$$

Entropy:

$$s = c_v \ln(p/\rho^\gamma) + \text{const} \qquad (1.9)$$

c_v is specific heat at constant density, γ is adiabatic index (usually = 5/3)

Energy loss function:

$$\mathcal{L} = \nabla\cdot\mathbf{q} - \frac{1}{\sigma}j^2 - \rho\nu\left\{\frac{1}{2}\sum_{j,k=1}^{3}\left(\frac{\partial v_j}{\partial x_k}+\frac{\partial v_k}{\partial x_j}\right)^2 - \frac{2}{3}(\nabla\cdot\mathbf{v})^2\right\} \qquad (1.10)$$

Governing equations

Force-balance in magnetised plasmas

A magnetic field in a conducting fluid exerts a force per unit volume F_{mag}

$$\vec{F}_{mag} = \vec{j} \times \vec{B} = \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_o}$$

where j is the current and B the magnetic induction (often referred to as magnetic field strength). This is the   on the particles.

The equation of motion of an element of material inside a 'flux tube' in a conducting fluid is

$$-\nabla p + \rho \vec{g} + \nabla . \overline{S} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_o} = \rho \dot{v}$$

where g is the local gravitational acceleration, p the gas pressure, ρ the density and S a tensor describing viscous stresses.

Setting $\dot{v} = 0$ we have the equation of magnetohydrostatic equilibrium.

Governing equations

Force-balance in magnetised plasmas

A magnetic field in a conducting fluid exerts a force per unit volume F_{mag}

$$\vec{F}_{mag} = \vec{j} \times \vec{B} = \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_o}$$

where j is the current and B the magnetic induction (often referred to as magnetic field strength). This is the sum of Lorentz forces on the particles.

The equation of motion of an element of material inside a ‘flux tube’ in a conducting fluid is

$$-\nabla p + \rho \vec{g} + \nabla . \overline{S} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_o} = \rho \dot{v}$$

where g is the local gravitational acceleration, p the gas pressure, ρ the density and S a tensor describing viscous stresses.

Setting $\dot{v} = 0$ we have the equation of magnetohydrostatic equilibrium.

MHS equilibrium

Hydrostatic pressure balance

Suppose: uniform vertical magnetic field

$$\mathbf{B} = B_0\mathbf{z}, \quad \mathbf{g} = -g\mathbf{z} \quad \Longrightarrow \quad \mathbf{j} = \mathbf{0}$$

i.e., no Lorenz force

MHS equation becomes

$$\frac{dp}{dz} = -\rho(z)g = \frac{g\tilde{\mu}}{RT(z)}p(z) = -\frac{p(z)}{\Lambda(z)}, \quad \text{where} \quad \underbrace{\Lambda(z) = \frac{RT(z)}{\tilde{\mu}g}}$$

Separate variables

$$\frac{dp}{p} = -\frac{1}{\Lambda(z)}, \Rightarrow \qquad \text{where} \quad \underbrace{n(z) = \int_0^z \frac{1}{\Lambda(u)}du}_{\text{integrated number of scale heights}}$$

MHS equilibrium

Hydrostatic pressure balance

Suppose: uniform vertical magnetic field

$$\mathbf{B} = B_0 \mathbf{z}, \quad \mathbf{g} = -g\mathbf{z} \quad \Longrightarrow \quad \mathbf{j} = \mathbf{0}$$

i.e., no Lorenz force

MHS equation becomes

$$\frac{dp}{dz} = -\rho(z)g = \frac{g\tilde{\mu}}{RT(z)}p(z) = -\frac{p(z)}{\Lambda(z)}, \quad \text{where} \quad \underbrace{\Lambda(z) = \frac{RT(z)}{\tilde{\mu}g}}_{\text{pressure scale height}}$$

Separate variables

$$\frac{dp}{p} = -\frac{1}{\Lambda(z)}, \Rightarrow \log p = -n(z) + \log p(0), \quad \text{where} \quad \underbrace{n(z) = \int_0^z \frac{1}{\Lambda(u)}du}_{\text{integrated number of scale heights}}$$

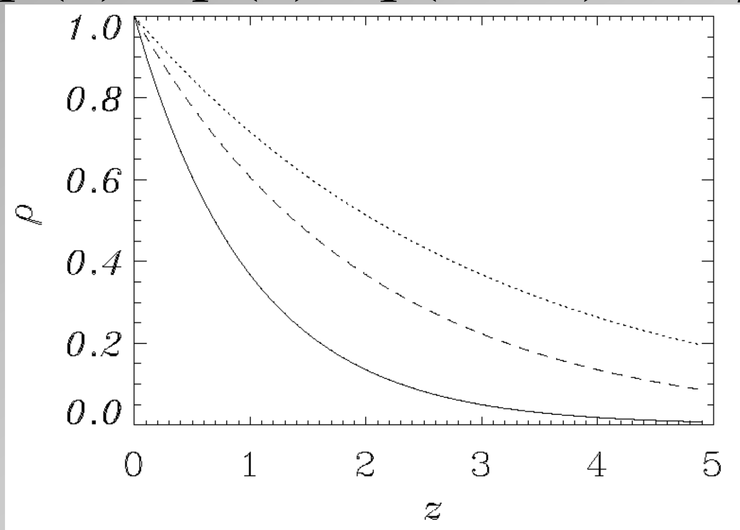
MHS equilibrium

Hydrostatic pressure balance

Solution $$p(z) = p(0)\exp[-n(z)]$$

Isothermal atmosphere (i.e. T, and Λ are const)

$$p(z) = p(0)\exp(-z/\Lambda) \qquad \rho(z) = \rho(0)\exp(-z/\Lambda)$$



Task: Mark the curves of Λ=1,2 and 3

Image: V. Nakariakov

www.warwick.ac.uk/go/space/

MHS equilibrium

Hydrostatic pressure balance

Typical examples (R=8.3x10^3 J K^{-1} mol^{-1})

Photosphere (g=274 m/s; μ=1.3, T=6000)

$$\Lambda = \frac{RT}{\tilde{\mu} g} = 140 \text{ km}$$

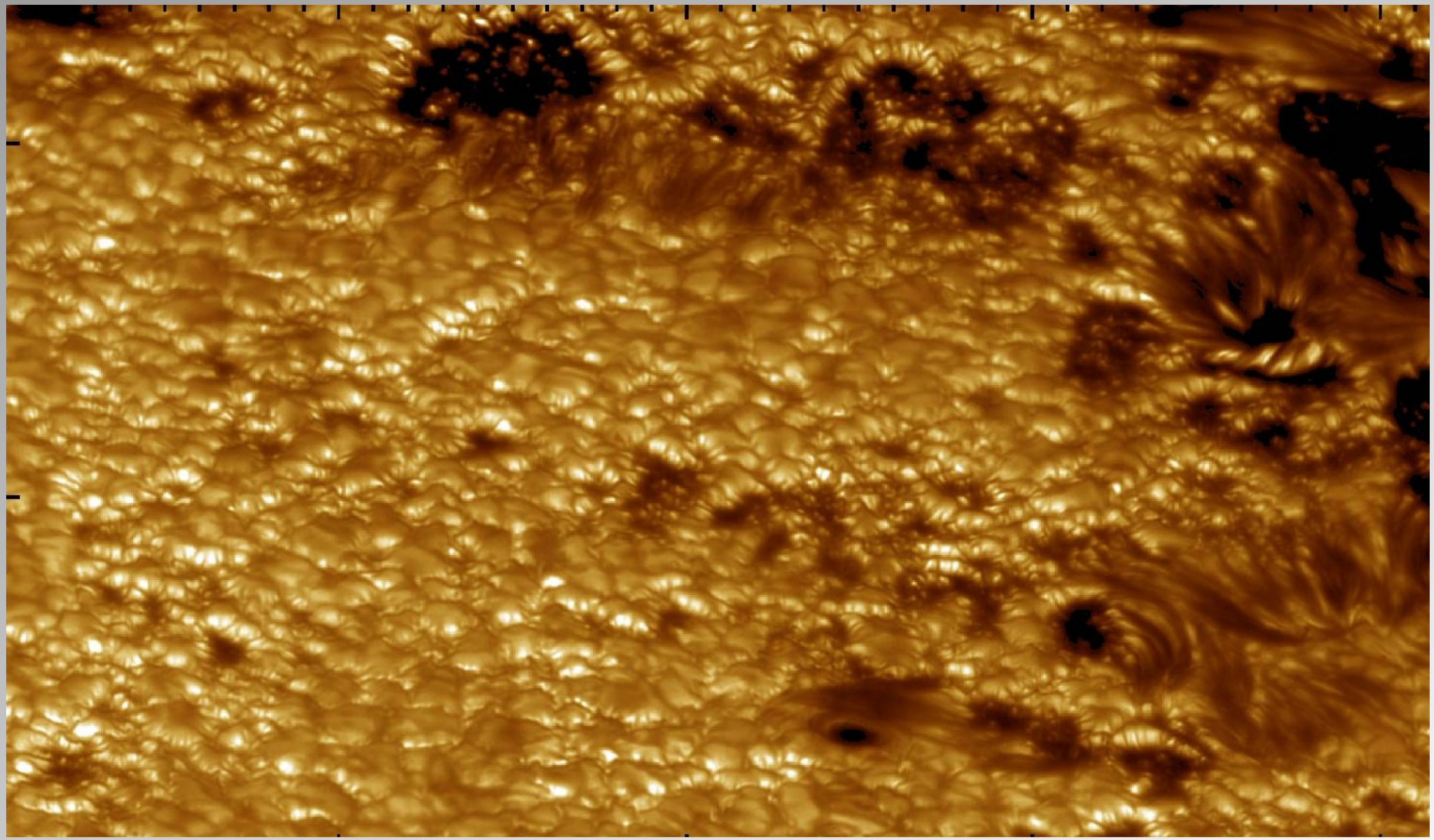
Corona (g=274 m/s; μ=0.6, T>10^6)

$$\underbrace{\Lambda \approx 50.5T \text{ m} = 50.5T[\text{MK}] \text{ Mm}}_{\text{Comapre to loop size!}}$$

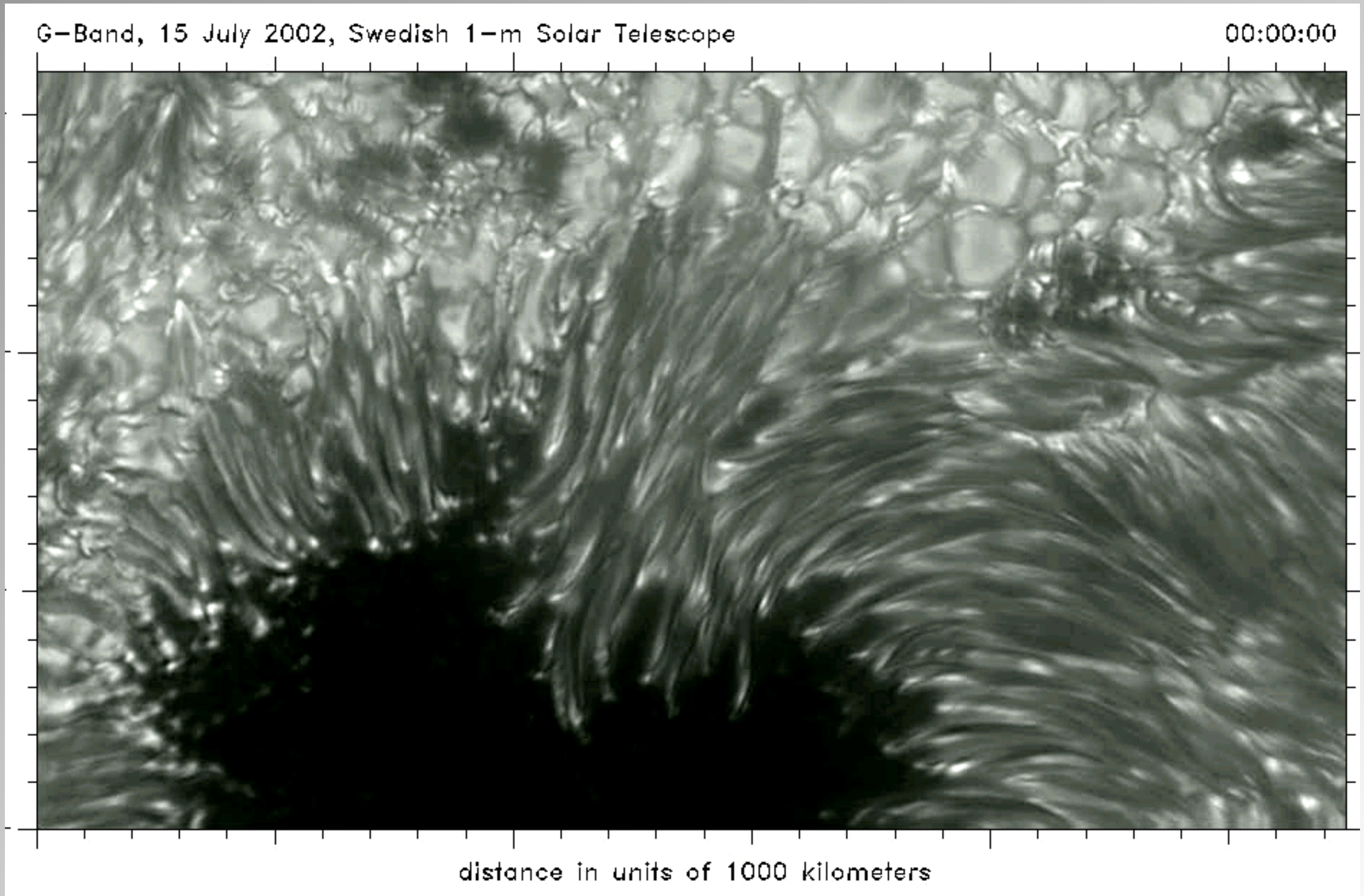
Earth's atmosphere (g=9.81 m/s; μ=029, T=300)

$$\Lambda = 8.7 \text{ km}$$

Photosphere: structure of sunspots

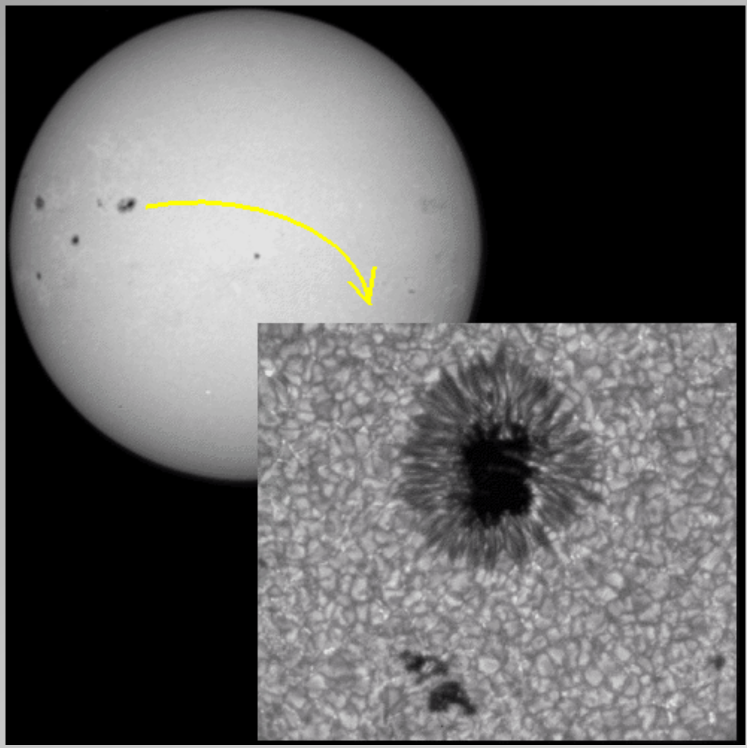


Photosphere: structure of sunspots



Photosphere: structure of sunspots

Sunspots are cooler than their surroundings because their strong magnetic field inhibits convection below the level of the photosphere. Hence, internal heat flux F_i, is reduced compared to external heat flux F_e



Sunspot field structure determined by lateral pressure balance

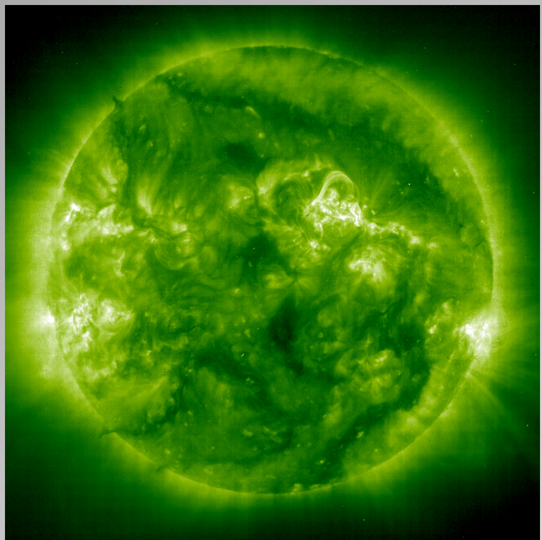
$$P_i + \frac{B_i^{\,2}}{2\mu_o} = P_e + \frac{B_e^{\,2}}{2\mu_o}$$

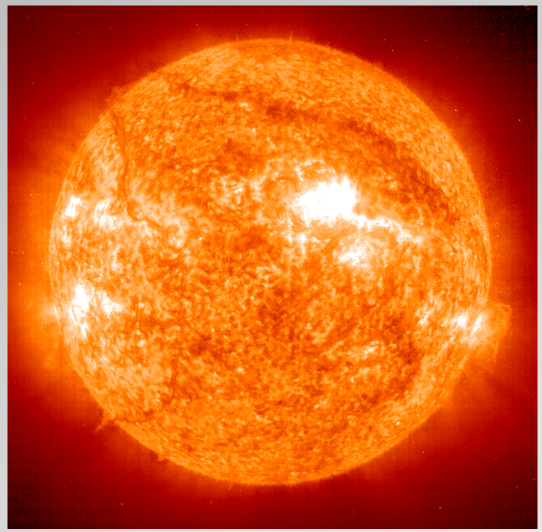
Prominences/filaments

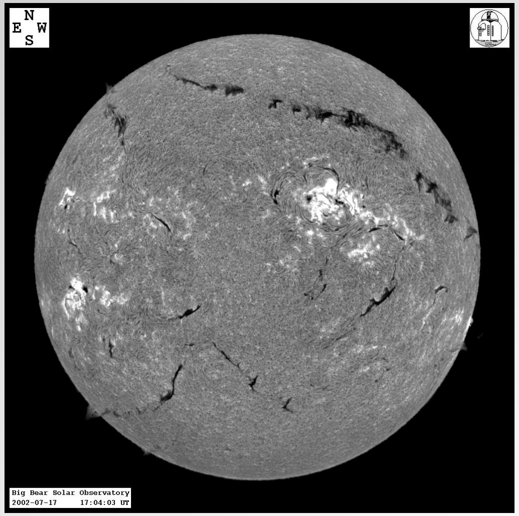
Filaments - called prominences when they appear in emission at the limb - are cool (20,000K) dense ($10^{21}m^{-3}$) gas which is thermally isolated from the surrounding corona.

They appear in active regions and in the quiet sun, and overlay magnetic neutral lines.

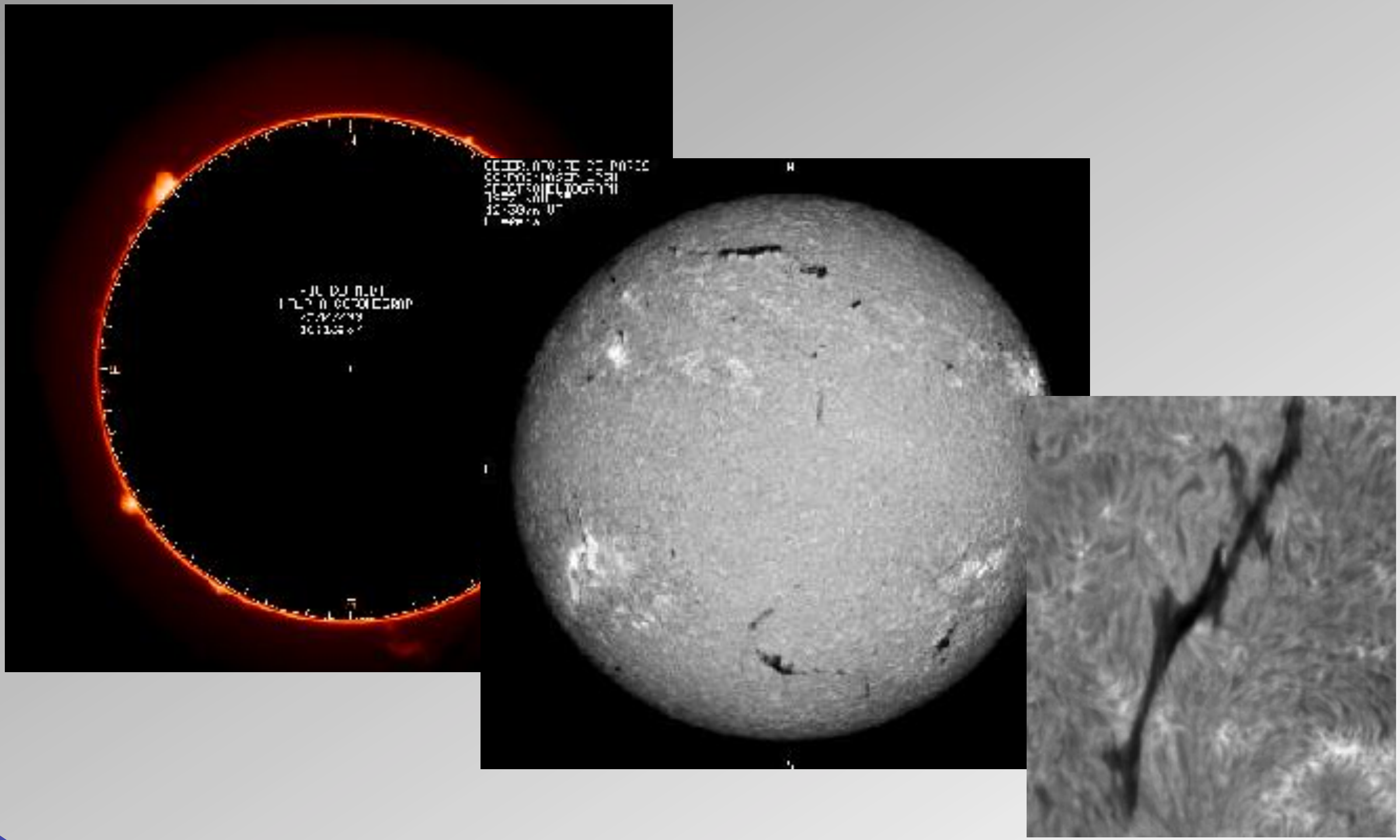
AR filaments tend to erupt within a few days, QS filaments can last and grow for weeks.







Prominences/filaments



Prominences/filaments

Filament support comes from the magnetic tension force in dipped magnetic fields or flux ropes. This opposes the downwards force of gravity.

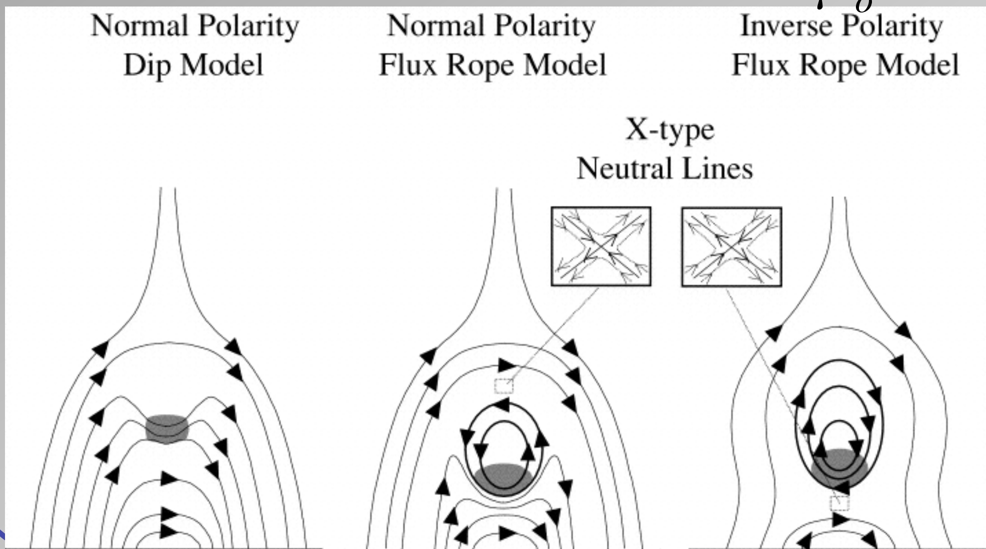
In static equilibrium (neglecting pressure gradients and viscous terms), the force-balance equation is:

$$\rho\vec{g} + \frac{(\nabla\times\vec{B})\times\vec{B}}{\mu_o} = \rho\vec{g} - \nabla\left(\frac{B^2}{2\mu_o}\right) + \frac{(\vec{B}.\nabla)\vec{B}}{\mu_o} = 0$$

Magnetic pressure

Magnetic tension

Image from Gilbert et al. 2001



These upward-curving field lines can be envisaged in a number of geometries.

Governing equations

Force-free and non-force-free fields

In the case of where all forces are negligible, except for the ***j*** x ***B*** force, the MHS equation reduces to

$$\vec{j} \times \vec{B} = 0$$

This is known as the 'force-free' condition. The gas pressure has virtually no influence (low-β plasma).

The photosphere is not force free. Moving outwards in the atmosphere the gas pressure and viscosity decrease, and the force-free condition becomes a good approximation (from ~500km above $\tau_{500\text{nm}} = 1$)

Above a few tenths of a solar radius, the field is again not force-free.

Governing equations

Force-free and non-force-free fields

In the case of where all forces are negligible, except for the ***j*** x ***B*** force, the MHS equation reduces to

$$\vec{j} \times \vec{B} = 0$$

This is known as the 'force-free' condition. The gas pressure has virtually no influence (low-β plasma).

Note: There are no cross-field currents in a force-free plasma. All currents are field-aligned.

The photosphere is not force free. Moving outwards in the atmosphere the gas pressure and viscosity decrease, and the force-free condition becomes a good approximation (from ~500km above $\tau_{500nm} = 1$)

Above a few tenths of a solar radius, the field is again not force-free.

Limits of applicability

- Speeds are much less than the speed of light.

(In the solar corona: v < a few thousand km/s).

- Characteristic times are much longer than the Larmour rotation period and the plasma period.

In the solar corona: f_{MHD} Hz. E.g., for B=10 G, n_e=5x10^{14} m^{-3}

f_{gyro}= Hz

f_{plasma}=

Limits of applicability

• Speeds are much less than the speed of light.
(In the solar corona: $v <$ a few thousand km/s).

• Characteristic times are much longer than the Larmour rotation period and the plasma period.

In the solar corona: $f_{MHD} < 1$ Hz. E.g., for B=10 G, n_e=5x10^{14} m^{-3}

$f_{gyro}=$ Hz

$f_{plasma}=$

Limits of applicability

- Speeds are much less than the speed of light.

(In the solar corona: v < a few thousand km/s).

- Characteristic times are much longer than the Larmour rotation period and the plasma period.

In the solar corona: $f_{MHD} < 1$ Hz. E.g., for B=10 G, n_e=5x10^{14} m^{-3}

f_{gyro}=1.52x10^{3}xB(G)= 1.52x10^4 Hz

f_{plasma}=9x $n_e^{1/2}$ (m^{-3})= 2x10^8 Hz

Limits of applicability

• The Hall effect is insignificant. The ratio of the dispersive and "wave" terms in the dispersion relation (the dispersive correction to the phase speed):

$$H = \frac{2\pi v_A}{\omega_{\text{gyro}} L} \approx \frac{5\times 10^2}{L[\text{m}]}$$

For $\lambda=5\text{x}10^5$ m H=

• Characteristic times are much longer than the collision times.

• Characteristic spatial scales are larger than the mean free path length

$$L >> l_{ii}[\text{m}] \approx \frac{7.2\times 10^7 T^2[\text{K}]}{n[\text{m}^{-3}]}$$

Limits of applicability

• The Hall effect is insignificant. The ratio of the dispersive and "wave" terms in the dispersion relation (the dispersive correction to the phase speed):

$$H = \frac{2\pi v_A}{\omega_{\mathrm{gyro}} L} \approx \frac{5\times 10^2}{L[\mathrm{m}]}$$

For $\lambda=5\mathrm{x}10^5$ m H=0.001

• Characteristic times are much longer than the collision times.

• Characteristic spatial scales are larger than the mean free path length

$$L >> l_{ii}[\mathrm{m}] \approx \frac{7.2\times 10^7 T^2[\mathrm{K}]}{n[\mathrm{m}^{-3}]}$$

Limits of applicability

• The Hall effect is insignificant. The ratio of the dispersive and "wave" terms in the dispersion relation (the dispersive correction to the phase speed):

$$H = \frac{2\pi v_A}{\omega_{\mathrm{gyro}} L} \approx \frac{5\times 10^2}{L[\mathrm{m}]}$$

For $\lambda = 5\mathrm{x}10^5$ m H=0.001

• Characteristic times are much longer than the collision times.

• Characteristic spatial scales are larger than the mean free path length

$$L >> l_{ii}[\mathrm{m}] \approx \frac{7.2\times 10^7 T^2[\mathrm{K}]}{n[\mathrm{m}^{-3}]} \qquad l_{ii} \approx 10^5 - 10^6 \text{ m}$$

Important limits

- Cold plasma

- Incompressible plasma

Important limits

- Cold plasma

ß=0

- Incompressible plasma

Important limits

- Cold plasma

$$ß=0$$

- Incompressible plasma

$$\gamma \to \infty$$

Frozen-in fields

The magnetic field is for the most part 'frozen-in' to the coronal plasma. This is the same as saying that the plasma is highly conducting.

We can demonstrate this by looking at field advection and diffusion. Start with Ohm's law:

$$\vec{E} + \vec{v} \times \vec{B} = \frac{\vec{j}}{\sigma} \qquad \sigma = \text{conductivity} = 1/\eta\mu_o$$

Take the curl of this equation, and use $\nabla \times \vec{E} = -(\partial\vec{B}/\partial t)$ to eliminate E.

$$\frac{\partial\vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta\nabla \times (\nabla \times \vec{B})$$

where we have also used $\nabla \times \vec{B} = \mu_0 \vec{j}$

Expanding the last term, and using $\nabla . \vec{B} = 0$ we arrive at the induction equation

$$\frac{\partial\vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta\nabla^2\vec{B}$$

Frozen-in fields

The two terms on the left hand side represent the advection of field by the flow, and the dissipation of field due to resistivity

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}$$

1 2

Normally in the solar atmosphere (e.g. corona), conductivity σ is very high, so $\eta = 1/\mu_o \sigma$ is very small.
In this case, term 2 is negligible in comparison with term 1. So the equation becomes

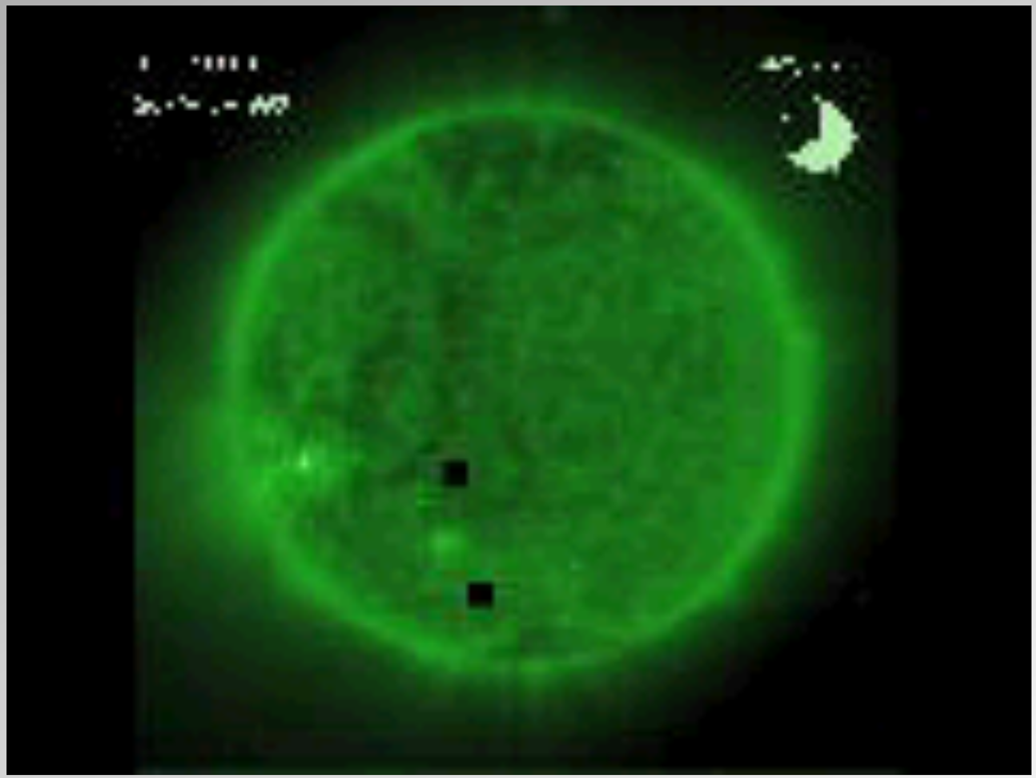
$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

v x ***B*** is the component of flow perpendicular to the magnetic field. So perpendicular flows distort B, and vice versa. **The field is locked to the plasma.**

(In fact one must prove that the total magnetic flux through a surface remains constant as the field is deformed).

Potential and force-free fields

Construct magnetic structure



Potential and force-free fields

If magnetic field $\boldsymbol{B}=(B_x, B_y, B_z)$ is known as a fnc of position, then the field lines are defined by

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B}$$

In parametric form in terms of the parameter s, the field lines satisfy

$$\frac{dx}{ds} = \frac{B_x}{B}, \qquad \frac{dy}{ds} = \frac{B_y}{B}, \qquad \frac{dz}{ds} = \frac{B_z}{B},$$

where the parameter s is the distance along the field line.

Potential and force-free fields

Example: $\boldsymbol{B}=B_0(y/a, x/a, 0)$, where B_0 and a are cnst

$$\frac{dx}{y/a} = \frac{dy}{x/a} \quad \Rightarrow \quad x\,dx = y\,dy \quad \Rightarrow \quad x^2 - y^2 = \pm c^2 = conts$$

Field lines: hyperbolae

This is a neutral point, or X-point

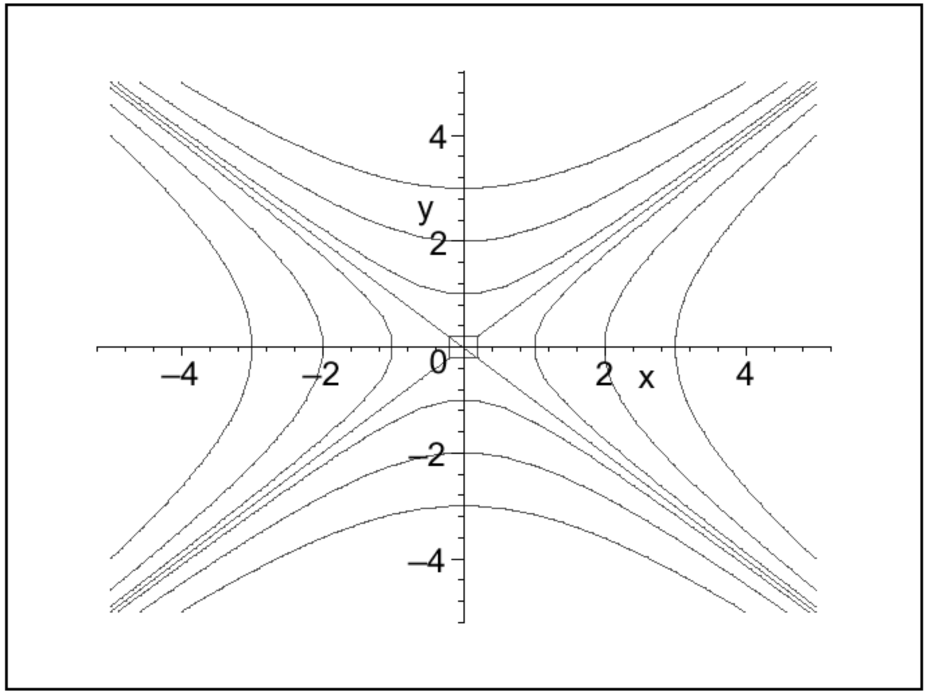


Image: V. Nakariakov

www.warwick.ac.uk/go/space/

Potential and force-free fields

Plasma-*β*

If characteristic length (L) << scale height (Λ) ➔ neglect gravity

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} \qquad \mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

$$\nabla p \approx \frac{p}{L} \qquad \frac{1}{\mu} \nabla \times \mathbf{B} \times \mathbf{B} \approx \frac{B^2}{\mu L}$$

Ratio of pressure gradient and Lorenz force:

$$\beta := \frac{\text{gas pressure}}{\text{magnetic pressure}} = \frac{p}{B^2 / 2\mu}$$

Potential and force-free fields

Plasma-β

Can be evaluated by the formula

$$\beta = 3.5x10^{-21}\, nTB^{2}, \text{ where } n\ [\mathrm{m}^{3}],\ T\ [\mathrm{K}], \text{ and } B\ [\mathrm{G}].$$

Solar corona:

Solar photospheric magnetic flux tubes:

Solar wind near Earth:

Potential and force-free fields

Plasma-*β*

Can be evaluated by the formula

$$\beta = 3.5x10^{-21} nTB^{2}, \text{ where } n \text{ [m}^{3}\text{]}, T \text{ [K], and } B \text{ [G].}$$

Solar corona: T=10^6 K, n= 10^{14} m^{-3}, B=10 G ➔ β=$3.5\text{x}10^{-3}$

Solar photospheric magnetic flux tubes:

Solar wind near Earth:

Potential and force-free fields

Plasma-β

Can be evaluated by the formula

$$\beta = 3.5x10^{-21}nTB^{-2}$$, where n [m^{-3}], T [K], and B [G].

Solar corona: T=10^6 K, n= 10^{14} m^{-3}, B=10 G ➔ β=3.5×10^{-3}

Solar photospheric magnetic flux tubes: T=6×10^3 K, n= 10^{23} m^{-3}, B=1000 G ➔ β=2

Solar wind near Earth:

Potential and force-free fields

Plasma-*β*

Can be evaluated by the formula

$$\beta = 3.5x10^{-21} nTB^{2}, \text{ where } n \text{ [m}^{3}\text{], } T \text{ [K], and } B \text{ [G].}$$

Solar corona: T=10^6 K, n= 10^{14} m^{-3}, B=10 G ➔ β=3.5x10^{-3}

Solar photospheric magnetic flux tubes: T=6x10^3 K, n= 10^{23} m^{-3}, B=1000 G ➔ β=2

Solar wind near Earth: T=2x10^5 K, n= 10^7 m^{-3}, B=6x10^{-5} G ➔ β=2

Potential and force-free fields

Force-free fields

If β<<1, gas pressure neglected w.r.t. magnetic pressure

$$0 \approx \mathbf{j} \times \mathbf{B}$$

Magnetic field called **force-free**

Potential force-free fields

Suppose **j**:=0

$$\mathbf{j} = \nabla \times \mathbf{B} = 0$$

Magnetic field called **potential**

Most general solution:

$\mathbf{B} = \nabla\varphi$, where φ is the scalar magnetic potential

Solenoidal condition ($\nabla \cdot \mathbf{B}=0$) has to be satisfied, i.e.

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Laplace equation

Potential and force-free fields

Potential fields

Solution: 2-dimensional plane [*xz*], method of separataion of variables φ:=*X(x)Y(y)*

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 = \text{const}$$

$$X'' = -k^2 X \Rightarrow X(x) =$$

$$Y'' = k^2 Y \Rightarrow Y(y) =$$

Boundary conditions: $\varphi(x,0)=F(x)$, $\varphi(0,y)= \varphi(l,y)=0$, $\varphi \to 0$ as $y \to \infty$, giving

$b=d=0$ and $\sin kl = 0 \Rightarrow k = \frac{n\pi}{l}$

Potential and force-free fields

Potential fields

Solution: 2-dimensional plane [*xz*], method of separataion of variables φ:=*X(x)Y(y)*

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 = \text{const}$$

$$X'' = -k^2 X \Rightarrow X(x) = a \sin kx + b \cos kx$$

$$Y'' = k^2 Y \Rightarrow Y(y) =$$

Boundary conditions: $\varphi(x,0)=F(x)$, $\varphi(0,y)= \varphi(l,y)=0$, $\varphi \to 0$ as $y \to \infty$, giving

$b=d=0$ and $\sin kl = 0 \Rightarrow k = \frac{n\pi}{l}$

Potential and force-free fields

Potential fields

Solution: 2-dimensional plane [*xz*], method of separataion of variables φ:=*X(x)Y(y)*

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2 = \text{const}$$

$$X'' = -k^2 X \Rightarrow X(x) = a \sin kx + b \cos kx$$

$$Y'' = k^2 Y \Rightarrow Y(y) = c \exp(-ky) + d \exp(ky)$$

Boundary conditions: $\varphi(x,0)=F(x)$, $\varphi(0,y)=\varphi(l,y)=0$, $\varphi \to 0$ as $y \to \infty$, giving

$b=d=0$ and $\sin kl = 0 \Rightarrow k = \frac{n\pi}{l}$

Potential and force-free fields

Potential fields

Full solution obtained by summing over all possible solutions. Let A_k=ac,

$$\varphi(x, y) = \sum_k A_k \sin kx \exp(-ky) \quad \text{where} \quad F(x) = \sum_k A_k \sin kx$$

Example

$$F(x) := \sin \frac{\pi x}{l} \Rightarrow A_1 = 1, A_n = 0, n \geq 2$$

$$\varphi(x, y) = \sin \frac{\pi x}{l} \exp(-\pi y / l)$$

$$\mathbf{B} = \nabla\varphi \qquad B_x = \frac{\partial\varphi}{\partial x} = B_0 \cos \frac{\pi x}{l} \exp(-\pi y / l),$$

$$B_y = \frac{\partial\varphi}{\partial y} = -B_0 \sin \frac{\pi x}{l} \exp(-\pi y / l)$$

Potential and force-free fields

Potential fields

Model for magnetic field lines in coronal arcades

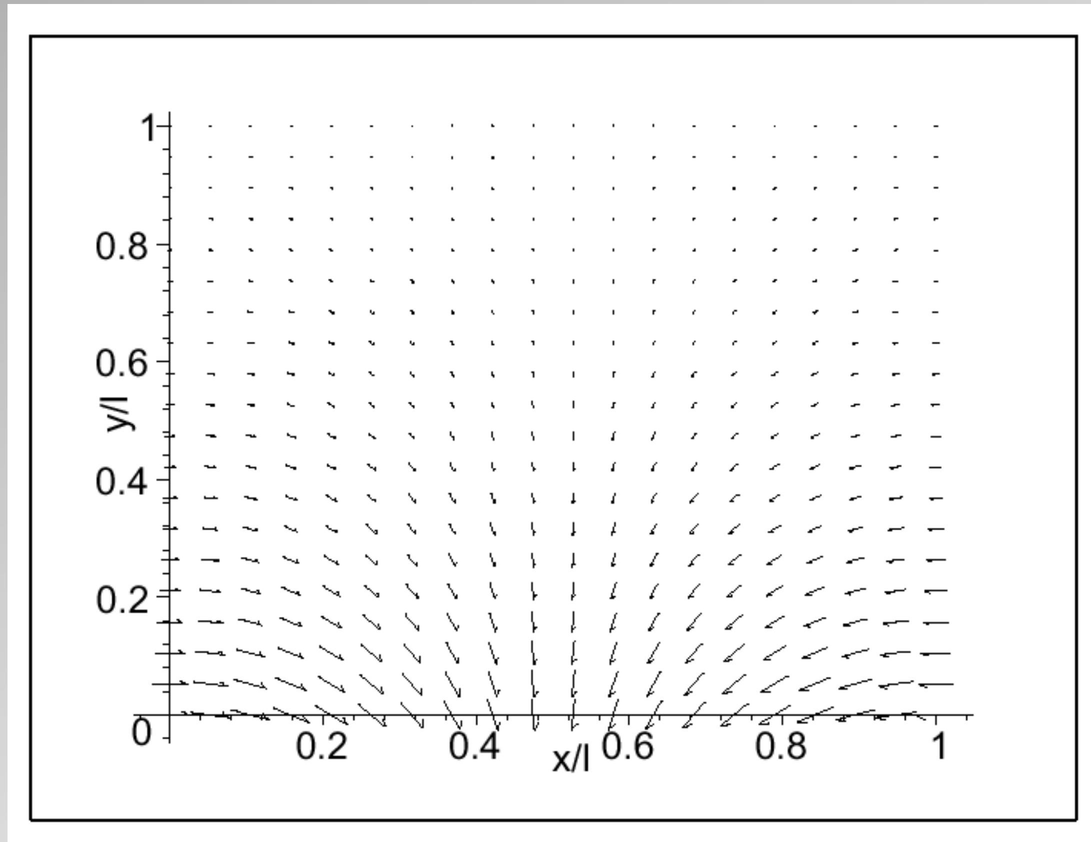


Image: V. Nakariakov

www.warwick.ac.uk/go/space/

Potential and force-free fields

Non-potential force-free fields

Suppose β<<1 and $L << \Lambda$ but $\boldsymbol{j}\neq 0$ from

$$0 \approx \mathbf{j} \times \mathbf{B} \Rightarrow \mu \mathbf{j} = \alpha \mathbf{B}$$ (current parallel to magnetic field)

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \alpha = \alpha(\mathbf{r}, t)$$

Constrains on α

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{\text{identically}=0} = \nabla \cdot (\alpha \mathbf{B})$$

$$0 = \nabla \cdot (\alpha \mathbf{B}) = \alpha \underbrace{\nabla \cdot \mathbf{B}}_{0} + \mathbf{B} \cdot \nabla \alpha$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$ (α=const along magnetic field lines!)

Potential and force-free fields

Non-potential force-free fields

- α:=0, force-free potential fields
- α:=const

$$\left.\begin{aligned} \nabla\times(\nabla\times\mathbf{B}) &= \nabla\times(\alpha\mathbf{B}) = \alpha\nabla\times\mathbf{B} = \alpha^2\mathbf{B} \\ \nabla\times(\nabla\times\mathbf{B}) &= \underbrace{\nabla(\nabla\cdot\mathbf{B})}_{=0} - \nabla^2\mathbf{B} = -\nabla^2\mathbf{B} \end{aligned}\right\} \quad -\nabla^2\mathbf{B} = \alpha^2\mathbf{B}$$

Helmholtz equation

- $\alpha:=\alpha(\mathbf{r})$

$$\left.\begin{aligned} \nabla\times(\nabla\times\mathbf{B}) &= \nabla\times(\alpha\mathbf{B}) = \alpha\nabla\times\mathbf{B} + \nabla\alpha\times\mathbf{B} = \alpha^2\mathbf{B} + \nabla\alpha\times\mathbf{B} \\ \nabla\times(\nabla\times\mathbf{B}) &= -\nabla^2\mathbf{B} \end{aligned}\right\}$$

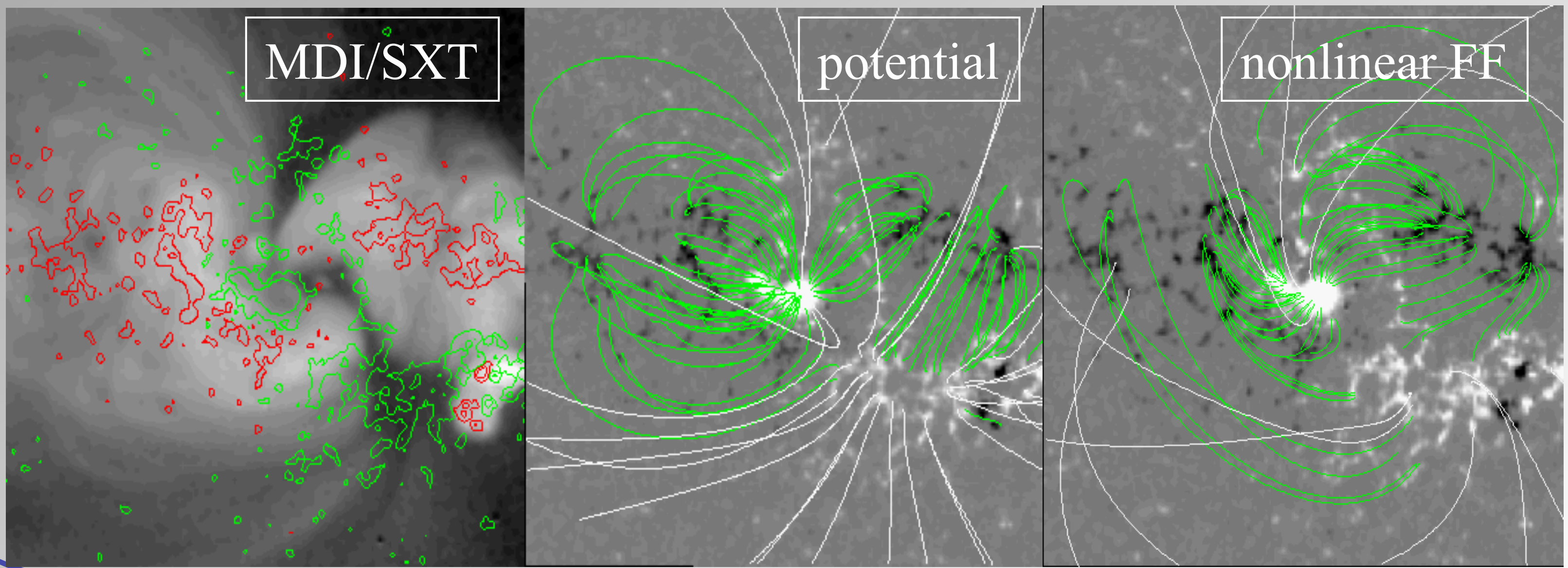
$$\alpha^2\mathbf{B} + \nabla^2\mathbf{B} = \mathbf{B}\times\nabla\alpha \qquad [\mathbf{B}\cdot\nabla\alpha = 0]$$

Photospheric magnetic fields

Force-free and non-force-free fields

From the same photospheric field distributions, one can extrapolate the coronal magnetic field (by solving $\nabla . B=0$ and $\nabla \times B=\alpha B$, with appropriate upper boundary conditions)

The extra energy stored in non-potential fields is exhibited as 'twist'. It is this excess of energy which can be released in the form of a solar flare or coronal mass ejection.



Equilibrium of coronal loops

The magnetic field has a lowest energy or 'potential' state, which is completely untwisted.

$$\nabla \times B = \mu_o j = 0$$

In general, a force-free magnetic field can carry field-aligned currents, which correspond to twisting up (putting energy into) the field. Generally we have

$$\nabla \times \vec{B} = \alpha(x, y)\vec{B}$$

$\alpha = 0$:	potential field. There are no currents
$\alpha =$ const:	linear force-free field. $j = \alpha B$
$\alpha \neq$ const:	non-linear FFF

Equilibrium of coronal loops

Loops must be heated by some mechanism(s), which may be varying in time and space. Locally, for equilibrium conditions, we have:

$$\dot{E}_{rad} + \dot{E}_{c} + \dot{E}_{heat} = 0$$

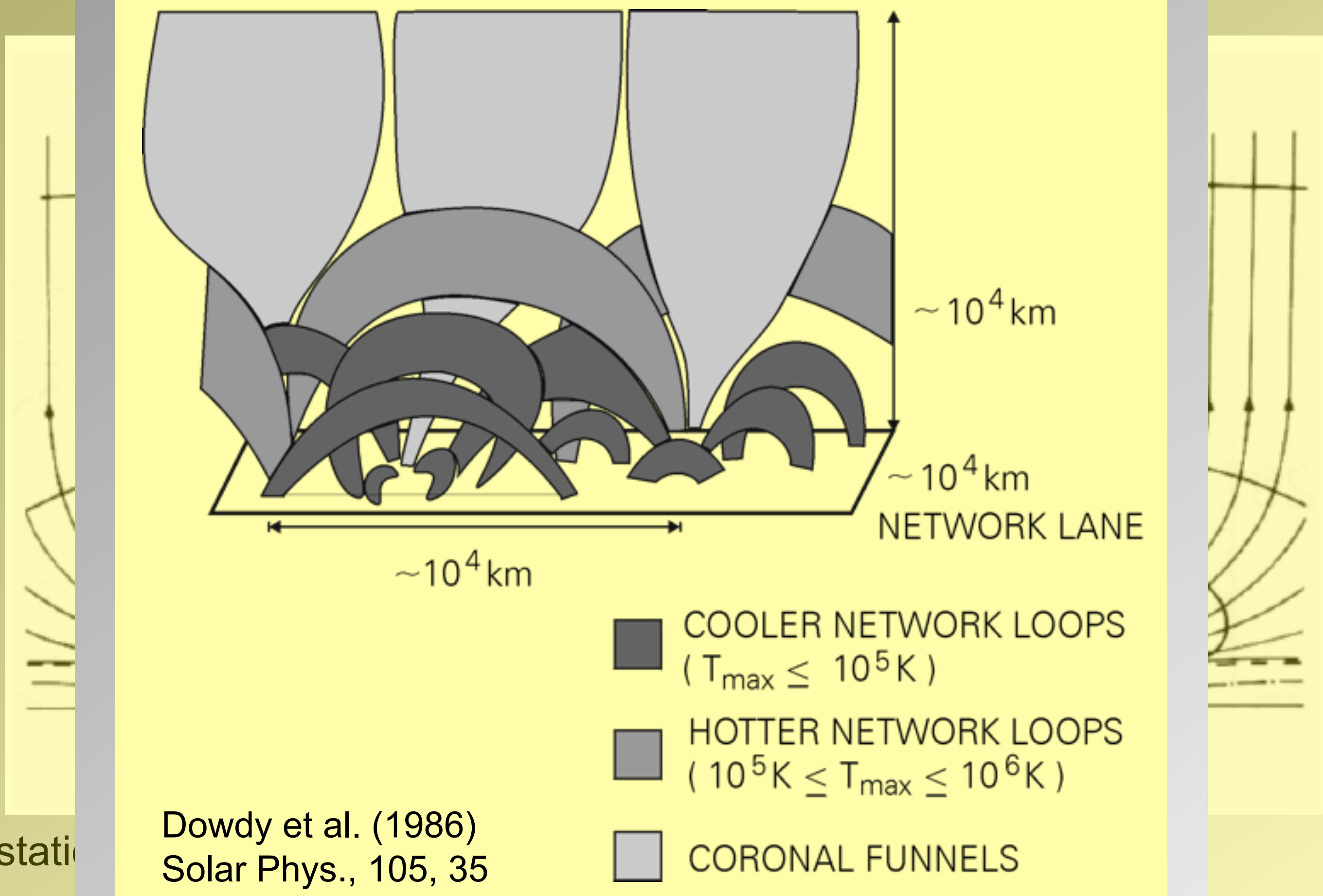
(The three main ideas are – basal heating, looptop heating, uniform heating. Observationally, basal heating looks slightly more likely than the others)

This equation is solved in tandem with the equation of hydrostatic equilibrium

$$\frac{dp}{ds} + \rho g = 0$$

With appropriate boundary conditions (conductive flux=0 at loop apex, monotonically increasing T with height, small vertical extent compared to coronal scale height) one obtains loop 'scaling laws', e.g.

$$T_{apex} \propto (pL)^{1/3}$$ ('RTV'= Rosner,Tucker Vaiana Law)



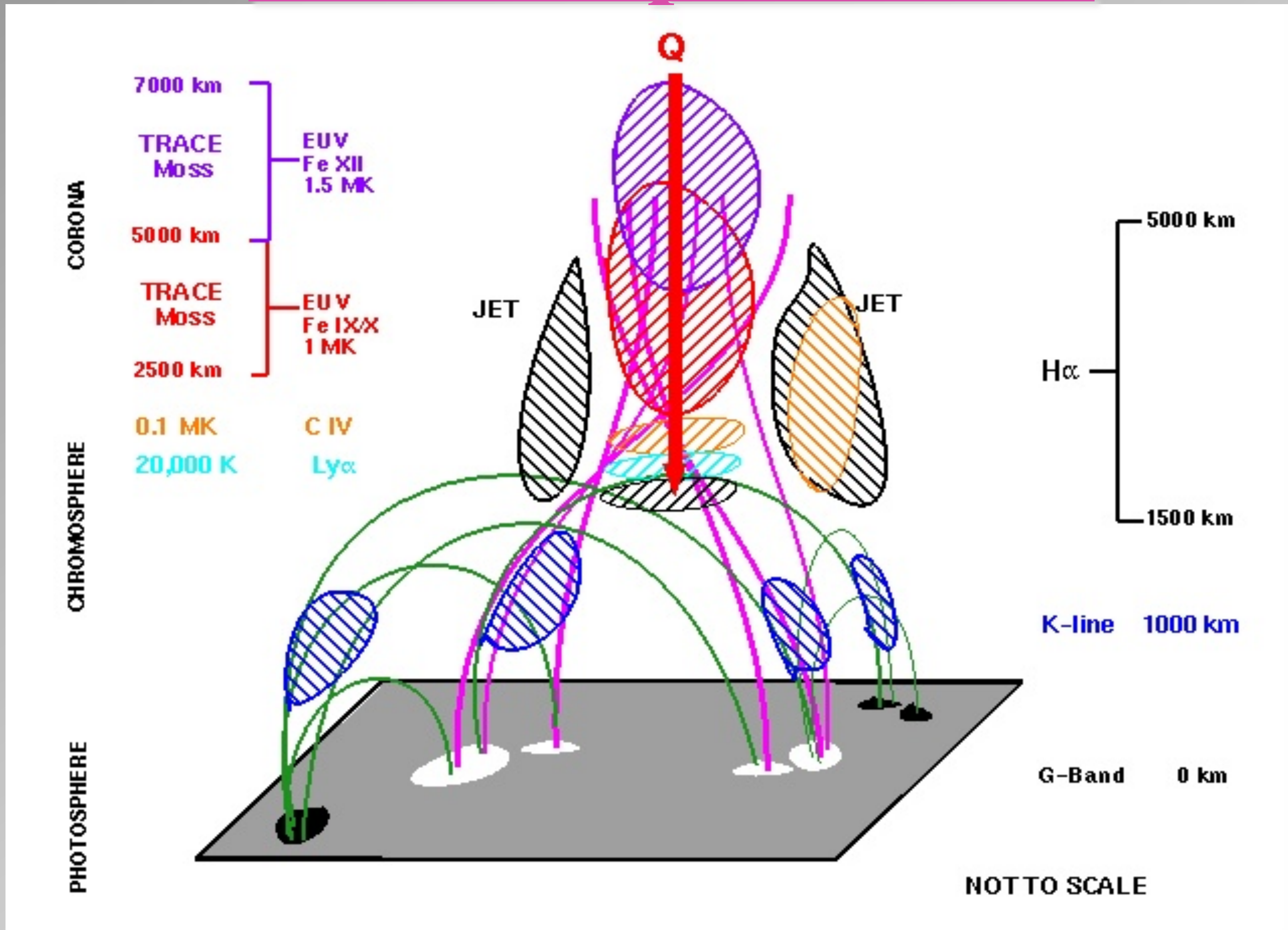
Dowdy et al. (1986)
Solar Phys., 105, 35

2D stati

- press

Gabriel (1976), Phil. Trans. A281, 339

Model improvement



(De Pontieu, Tarbell, Erdélyi, ApJ 590, 502, 2003)

The end